First Exam

1. Let $f(x, y, z) = (x + 2y)^2 + (y + 2z)^2 + (z + 2x)^2$ and consider the point P_0 of coordinates $(-2, 1, 2)$. Find the direction in which the function increases most rapidly (the answer should be a unit vector) and compute the magnitude of the greatest rate of increase.

Solution: We compute

$$\nabla f = (10x + 4y + 4z, 4x + 10y + 4z, 4x + 4y + 10z).$$

At the given point the gradient is $(-8, 10, 16)$. To normalize it we compute its magnitude: $\sqrt{64 + 100 + 256} = \sqrt{420} = 2\sqrt{105}$. The direction is

$$\left(\frac{-4}{\sqrt{105}}, \frac{5}{\sqrt{105}}, \frac{8}{\sqrt{105}}\right).$$

The value of the rate of change is the magnitude of the gradient, so it is $2\sqrt{105}$.

2. Let $\mathbf{R}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + (1-t^2)\mathbf{k}$.
(a) Find the unit tangent vector (do not compute the principal unit normal vector).
(b) Find the curvature of the curve at $t = 0$.

Solution: Using the product rule, we compute

$$\mathbf{R}'(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} - 2t\mathbf{k}$$

and so

$$\begin{aligned} \|\mathbf{R}'(t)\|^2 &= (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 4t^2 \\ &= \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 4t^2 \\ &= 1 + t^2 + 4t^2 = 5t^2. \end{aligned}$$

Consequently

$$\mathbf{T}(t) = \frac{\cos t - t\sin t}{\sqrt{1+5t^2}} + \frac{\sin t + t\cos t}{\sqrt{1+5t^2}} - \frac{2t}{\sqrt{1+5t^2}}.$$

Also

$$\mathbf{R}''(t) = (2\sin t - t\cos t)\mathbf{i} + (2\cos t + t\sin t)\mathbf{j} - 2\mathbf{k}.$$

Then $\mathbf{R}'(0) = \mathbf{i}$, $\mathbf{R}''(0) = 2\mathbf{j} - 2\mathbf{k}$, so $\mathbf{R}'(0) \times \mathbf{R}''(0) = 2\mathbf{k} + 2\mathbf{j}$. The norm of $\mathbf{R}'(0)$ is 1 and of $2\mathbf{k} + 2\mathbf{j}$ is $2\sqrt{2}$. So the curvature at 0 is $(2\sqrt{2})/1^3 = \sqrt{8} = 2\sqrt{2}$.

3. Find the critical points of $f(x, y) = 3xy^2 - 2x^2y + 36xy$ and classify them as a relative maximum, a relative minimum, or a saddle point.

Solution: To find the critical points we have to solve the system

$$\begin{aligned} f_x &= 3y^2 - 4xy + 36y = 0 \\ f_y &= 6xy - 2x^2 + 36x = 0, \end{aligned}$$

Factoring we obtain the system

$$\begin{aligned} y(3y - 4x + 36) &= 0 \\ x(6y - 2x + 36) &= 0. \end{aligned}$$

The first equation is equal to zero if and only if $y = 0$ or $3y - 4x + 36 = 0$, that is if $y = 0$ ir $y = \frac{4}{3}x - 12$. Substituting in the second equation and solving we obtain the pairs $(0, 0), (0, -12), (18, 0), (6, -4)$, which are the critical points $D = f_{xx}f_{yy} - (f_{xy})^2$ is negative for the first three and positive for the fourth, so $(0, 0), (0, -12), (18, 0)$ are saddle points. Because $f_{xx}(6, -4) > 0$ this is a local minimum.

4. Find f_{xy} if

$$f(x, y, z) = \frac{xy + yz}{xz}.$$

Solution: We have

$$f(x, y, z) = \frac{y}{z} + \frac{y}{x},$$

and

$$f_x = -\frac{y}{x^2}, \quad f_{xy} = -\frac{1}{x^2}.$$

5. Maximize $f(x, y) = 16 - x^2 - y^2$ subject to the constraint $x + 2y = 6$.

First solution: Using Lagrange multipliers, we set

$$\begin{aligned} -2x &= \lambda \\ -2y &= 2\lambda \\ x + 2y &= 6 \end{aligned}$$

so $y = 2x$ and consequently $x = 6/5$, $y = 12/5$ and the maximum is $44/5 = 8.8$.

Second solution: Setting $x = 6 - 2y$, we have to maximize the function $f(x, y) = 16 - (6 - 2y)^2 - y^2$. Its derivative is $24 - 8y - 2y$; setting it equal to zero gives $y = 12/5$, etc.