Final Exam

Each problem is worth the same number of points. You will be graded both on the correctness and on the clarity of the argument.

1. Let a, b and m be integers. Prove that if $2a + 3b \geq 12m + 1$, then $a \geq 3m + 1$ or $b \geq 2m + 1$.

2. Prove that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the equality

$$f(x - 1) + f(2 - x) = x$$

for all $x \in \mathbb{R}$.

3. Prove that $\sqrt{3} + \sqrt{5}$ is an irrational number.

4. Prove by induction that 7 divides $3^{2n} - 2^n$ for every nonnegative integer n.

5. Prove that among any $n + 1$ people selected from a group of n women and their husbands, there is a married couple.

6. Prove that for every $n \in \mathbb{N}$,

$$1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \cdots + n! \cdot n = (n + 1)! - 1.$$

Here $n!$ denotes the product of the numbers $1, 2, 3, ..., n$, so that $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$.

7. Show that among any six people one can either find three people who know one another, or three people so that none of them knows any of the other two.