The Quantum Invariant of the Complement of a Regular Neighborhood of a Link

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Abstract

In this paper we show how the formula for the $sl(2, C)$ quantum invariant of the complement of a regular neighborhood of a link can be proved in the context of topological quantum field theory with corners. As an application we will describe a short proof of the formulas for the colored Jones polynomials of torus knots.

0. INTRODUCTION

N. Reshetikhin and V. Turaev [RT] defined a smooth topological quantum field theory (shortly TQFT) for 3-manifolds, related to the Jones polynomial [Jo]. In [Ge], following previous work done in [Wa] and [FK], there has been constructed a theory with corners that underlies the one from [RT] . In such a TQFT one can recover the invariants of 3-manifolds from the basic data by using the axioms described in [Wa].

The construction from [Ge] relies on the decompositions of surfaces into disks, annuli and pairs of pants (called DAP-decompositions). More precisely, a DAP-decomposition consists of a collection of circles that cut the surface into elementary surfaces (disks, annuli and pairs of pants), an ordering of these elementary surfaces, and for each of them, a numbering of their boundary components and a family of arcs (called seams) connecting these boundary components, that keep track of the twistings. The modular functor is constructed by describing the vector spaces associated to elementary surfaces, and the morphisms associated to transformations of DAP-decompositions (called moves). Thus, instead of concentrating on the mapping class group, one deals with the groupoid of moves.

Like in the case of the mapping class group, one has to extend the groupoid of transformations by using the signature cocycle. However, unlike in the case of the mapping class group, this is not enough, since there appears a sign obstruction for some of the Moore-Seiberg equations. As we shall see below, this phenomenon is mainly due to an asymmetry of the skein relation of the Jones polynomial. In [Ge], in order to make the whole theory well defined, the structure of DAP-decompositions has been enriched by introducing a collection of simple closed curves (bands) on each elementary surface, parallel to the boundary components of this surface, indexed by the elements of the Klein four group. To simplify the formalism we have introduced an arrow notation, by putting on each curve an arrow pointing inwards for the element a of the Klein group, outwards for b, two arrows pointing in opposite directions for c or no arrow at all for the identity, in this latter case we usually do not draw the band anymore. Arrows are allowed to slide over decomposition circles and get multiplied.

In the present paper we intend to find some applications of this theory. First we show the presence of the skein relation of the Jones polynomial at the level of invariants of 3-manifolds with boundary, and explain how this makes the Klein group structure necessary. Then we prove the invariant formula for the complement of a regular neighborhood of a link. As one should expect, the coordinates of this invariant will be the colored Jones polynomials. As a consequence this will show how the Reshetikhin-Turaev invariant formula for closed manifolds can be deduced from axioms. Finally, we will give a short proof of the invariant formula for the quantum invariants of torus knots. Recall that formulas for the invariants of torus knots have been exhibited in [JR], but, as we shall see below, the TQFT with corners makes their computation extremely simple.

Before starting, let us recall some definitions and introduce notations that will be needed in the sequel. For a more extensive treatment of these facts we recommend [Wa], [FK] and [Ge]. Quantum invariants are always computed for some positive integer r, called level. For each integer n one defines its quantization to be $[n] = (t^{2n} - t^{-2n})/(t^2 - t^{-2})$ where $t = exp(\pi i/2r)$. Also let $X = \sqrt{[1]^2 + [2]^2 + \cdots + [r-1]^2}$.

In [RT] there is described a Hopf algebra U_t which is a quantum deformation of $sl(2, C)$, and together with it, for each integer n, $1 \leq n \leq r-1$, an irreducible n-dimensional representation denoted by $\underline{n}$. The set of numbers $1 \leq n \leq r-1$ will be used for the labeling of the boundaries of surfaces. For these representations a version of the Clebsch-Gordon formula holds, namely

$$\underline{m} \otimes \underline{n} = \bigoplus_p \underline{p} \oplus B$$

where the sum is over all $|m-n|+1 \le p \le min\{m+n-1, 2r-1-m-n\}$ with $m+n+p$ odd, and B is a representation on which all morphisms have quantum trace equal to 0. When we consider tensor products we will always factor by B.

In the sequel we will use the boldface writing for k-tuples of integers (i.e. $\mathbf{n} = (n_1, n_2, \cdots, n_k)$). We also let $[\mathbf{n}] = [n_1][n_2]\cdots[n_k]$.

Recall that for a diagram of a framed link with the blackboard framing, having k components, by labeling each of the components by an irreducible representation $\underline{n_i}$, $i = 1, 2, \cdots, k$, one can view the diagram as a morphism of the trivial representation. In this way we get a link invariant, namely the colored Jones polynomial $J(L, \mathbf{n})$ (denoted by $\lambda(L, \mathbf{n})$ in [RT]). If $n_i = 2$, $i = 1, 2, \cdots, k$, we will simply denote this polynomial by $J(L)$. Note that $J(L)$ is a version of the classical Jones [Jo] polynomial, which takes the framing into account. Whenever we mention the Jones polynomial we will actually refer to this normalized version of it.

For the modular functor, let us recall that the vector space assigned to a disk is $\mathbf{C}$ if the boundary of the disk is labeled by $\underline{0}$ and 0 otherwise. Also, the vector space assigned to an annulus with boundary components labeled by m is $V_m^m = \{\phi : \underline{m} \to \underline{m}, \phi$ module homomorphism$\}$. It is 1-dimensional and we chose a basis element β_m^m such that the map of contracting two neighboring annuli to one corresponds to $\beta_m^m \otimes \beta_m^m \to \beta_m^m$. One defines the pairing on V_m^m by $< \beta_m^m, \beta_m^m >= X/[m]$. For a pair of pants with boundary components labeled by p, m, n respectively, the vector space is $V_p^{mn} = \{\phi : \underline{m} \otimes \underline{n} \to \underline{p}$, ϕ module homomorphism$\}$. This space is either 0 or 1-dimensional, when it is 1-dimensional we let β_p^{mn} be a basis element that is an orthogonal projection with respect to the trace pairing. One defines the pairing $< \beta_p^{mn}, \beta_p^{nm} >= X^2/([m][n][p])^{1/2}$ The vector space of an arbitrary surface with labeled boundary is obtained according to the following gluing rule. If Σ is a surface endowed with a DAP-decomposition and a Klein group structure, C and C' two sets of boundary components of Σ, $g : C \to C'$ the parametrization reflecting map and Σ_g be Σ glued by g, then

$$V(\Sigma_g, l) = \bigoplus_{x \in \mathcal{L}(C)} V(\Sigma, (l, x, \hat{x}))$$

where the sum is over all labelings of C and l is any labeling of the boundary of Σ_g. We recall that one can glue two boundary components together only if they are both numbered by 2 or 3 or both by 1 and the product of the indices of their associated bands is a or b, or if one is numbered by 2 or 3 and the other by 1 and the product of the indices of their assocated bands is e or c. The move that adds an annulus to a DAP-decomposition will be called expansion by an annulus, its inverse will be called contraction by an annulus.

A surface with a DAP-decomposition and a Klein group structure will be called a completely extended surface (ce-surface). Consequently, a ce-3-manifold will be a triple consisting out of a 3-manifold, a DAP-decomposition and Klein group structure on its boundary, and an integer called framing. Note that if one only works with manifolds whose boundaries are tori, the Klein group structure is not necessary, thus in these situations we won't mention its existence. However, this structure will play an important role in making our first result consistent.

1. THE SKEIN RELATION FOR THE INVARIANTS OF 3-MANIFOLDS

In this section we will show the presence of the skein relation of the Jones polynomial at the level of 3-manifolds. Let us consider the three genus 2 ce-handlebodies described in Fig. 1, obtained by removing solid cylinders out of balls. In this figure the circled numbers define the ordering of the elementary surfaces, and the other numbes define the ordering of the boundary components of each elementary surface.

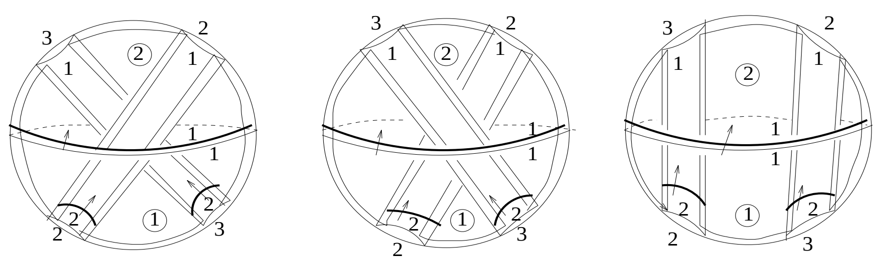

Figure 1.

Let $(M_1, D_1, 0)$, $(M_2, D_2, 0)$, and $(M_3, D_3, 0)$ be the three ce-3-manifolds obtained by gluing these handlebodies, along the punctured spheres on the exterior, via the same ce-morphism to the same ce-3-manifold. Note that the interior annuli will be included in the boundaries of M_1, M_2 and M_3 respectively.

From the construction of the modular functor we see that there is a subspace V_i of $V(\partial M_i)$ corresponding to the case where the boundary components of the two annuli are labeled by 2 (i.e. by the 2-dimensional irreducible representation). Moreover $V(\partial M_i)$ splits naturally as $V_i \oplus V_i'$, where V_i' is the subspace that corresponds to the labeling of the boundaries of the two annuli by labels different from 2, and $Z(M_i, 0)$ can be written as $v_i \oplus v_i'$ with $v_i \in V_i$ and $v_i' \in V_i'$. It is not hard to see that the subspaces V_1, V_2 and V_3 are canonically isomorphic, so we can consider that v_1, v_2 and v_3 are vectors in the same vector space.

Theorem 1.1. With the above notations the following identity holds:

$$tv_1 - t^{-1}v_2 = (t^2 - t^{-2})v_3 \tag{1}$$

Proof: The surfaces along which the gluing takes place are planar, so no change of framing occurs in the process of gluing. Therefore if we prove that the same relation holds for the invariants of the three handlebodies we are done.

Consider the first handlebody. Let us apply the move B_{23} in the upper hemisphere to change the seams and numberings of boundary components like in Fig. 2, then contract the annuli to get two pairs of pants glued together.

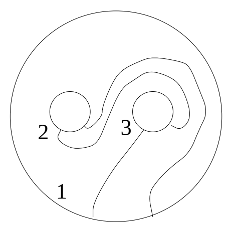

Figure 2.

The newly obtained ce-handlebody is the mapping cylinder of the map on a pair of pants that permutes the boundary components numbered 2 and 3. By the mapping cylinder axiom the invariant of the ce-handlebody should be the identity matrix in $(\bigoplus_{m,n,p} V_p^{mn})^* \otimes (\bigoplus_{m,n,p} V_p^{mn})$, so the invariant is $\oplus_{m,n,p}\hat{\beta}_p^{mn} \otimes \beta_p^{mn}$, where $\hat{\beta}_p^{mn}$ is the dual basis (thus $<\hat{\beta}_p^{mn}, \beta_p^{mn}>= 1$). To get the original DB-structure, we have to apply B_{23}^{-1} in the upper hemisphere and decontract the two annuli. The invariant in the new setting will be $\oplus_{m,n,p}\hat{\beta}_p^{mn} \otimes B_{23}^{-1}\beta_p^{mn} \otimes \beta_m^m \otimes \beta_n^n$. A similar argument shows that the invariants of the other two ce-handlebodies are $\oplus_{m,n,p}\hat{\beta}_p^{mn} \otimes B_{23}\beta_p^{mn} \otimes \beta_m^m \otimes \beta_n^n$ and $\oplus_{m,n,p}\hat{\beta}_p^{mn} \otimes \beta_p^{mn} \otimes \beta_m^m \otimes \beta_n^n$.

Let us recall ([KM]) that for the matrix $\check{R} : \underline{2} \otimes \underline{2} \to \underline{2} \otimes \underline{2}$ that corresponds to the positive crossing of two strands labeled by 2-dimensional representations, the following relation holds

$$t\check{R} - t^{-1}\check{R}^{-1} = (t^2 - t^{-2})Id$$

On the other hand $B_{23}^{-1}\beta_p^{mn} = \beta_p^{mn} \circ \check{R}$ and $B_{23}\beta_p^{mn} = \beta_p^{mn} \circ \check{R}^{-1}$, both as homomorphisms $\underline{m} \otimes \underline{n} \to \underline{p}$. Hence if we let $m = n = 2$ we get

$$tB_{23}^{-1}\beta_p^{mn} - t^{-1}B_{23}\beta_p^{mn} = (t^2 - t^{-2})Id$$

and the proposition is proved.□

Remark. In the absence of the Klein group structure one could not have such a result. Indeed, the relation we wrote for $\check{R}$ holds on oriented strands. On the other hand, the polynomial J (which we recall is computed by labeling all link components by $\underline{2}$ and using the $\check{R}$-matrix for the crossings) has the following asymetry of its skein relation. Let L_+, L_- and L_0 be three framed links (with the blackboard framing), that are identical except in a disk, where they look like in Fig. 3.a, then by Theorem 4.3 in [KM] we have

$tJ(L_+) - t^{-1}J(L_-) = (t^2 - t^{-2})J(L_0)$.

Now suppose that L_+ is a *knot*. If we resolve the crossing in the "wrong" way, getting the smoothing $\bar{L}_0$ (see Fig. 3.b), then the following relation holds

$tJ(L_-) - t^{-1}JL_+ = -(t^2 - t^{-2})J(\bar{L}_0)$.

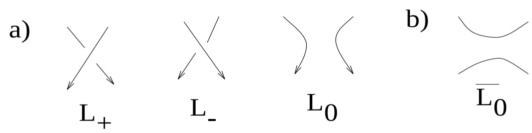

Figure 3.

Thus in our situation if we permute the numberings of the boundary components of the annuli we may run into a sign problem. However, as we see in [Ge], the permutation of these numbers produces also a modification of the Klein group structure, making the result consistent.

Let us show how this result can be used to prove the formula for the quantum invariant of the complement of the regular neighborhood of a link.

Proposition 1.2. Let L be a framed link with k components with the blackboard framing. Let N be a regular neighborhood of L and $M = S^3 - N$. Consider the DAP-decomposition and Klein group structure on ∂M such that each torus is decomposed into an annulus along a meridian of L, the seams coincide with the framing and the bands are all indexed by the identity element of the Klein group. Then

$$Z(M, D, 0) = \frac{1}{X} \sum_{\mathbf{n}=\mathbf{1}}^{\mathbf{r-1}} J(L, \mathbf{n}) \beta_{\mathbf{n}}^{\mathbf{n}} \tag{2}$$

where $\beta_{\mathbf{n}}^{\mathbf{n}} = \beta_{n_1}^{n_1} \otimes \beta_{n_2}^{n_2} \otimes \cdots \otimes \beta_{n_k}^{n_k}$.

Proof: Let us first prove it for the unknot. The complement of the unknot is a solid torus, and the DAP-decomposition is determined by a longitude of this solid torus (which is a meridinal circle for the knot). If we consider the DAP-decomposition obtained by taking another longitude of the torus parallel to the given one we get the mapping cylinder of an annulus, hence, by the mapping cylinder axiom the invariant of the solid torus will be $\sum_m \hat{\beta}_m^m \otimes \beta_m^m = \sum_m ([m]/X) \beta_m^m \otimes \beta_m^m$.

Contracting one of the annuli we get $\sum_m([m]/X)\beta_m^m$ which is the desired result since the colored Jones polynomial for the unknot colored by m is $[m]$.

By the gluing axiom, if we take the connected sum of two ce-3-manifolds, the invariant of the new ce-manifold is the product of the invariants of the previous two, multiplied by X. This shows that the formula holds for the trivial link with framing 0. To see that the formula also holds for the trivial link with arbitrary framing, let us note that if we change the framing of a link component by $m \in \mathbf{Z}$, then we have a move T^{-m} in the corresponding boundary component of the link complement, where T is the move that consists of the right handed twist on the torus. By the way T has been defined in [Ge], namely as multiplication by $t^{-n_i^2+1}$ where n_i is the label of the decomposition circle, we see that T^{-m} changes the coordinate of the invariant corresponding to the labeling of the link component by n_i by a factor of $t^{m(n_i^2-1)}$, which is exactly what happens to the link invariant, hence the formula holds in this case, too.

Now let L, M and D be like in the statement. Let $Z(M,D,0) = 1/X\sum c_{\mathbf{n}}\beta_{\mathbf{n}}^{\mathbf{n}}$. If $\mathbf{n} = (n_1, n_2, \cdots, n_k)$, with $n_i = 1$ or 2, then $c_{\mathbf{n}} = J(L,\mathbf{n})$. Indeed, in the computation of $J(L,\mathbf{n})$, the components labeled by 1 can be neglected. The same is true for the boundary components of M whose decomposition circles are labeled by 1, they can be removed by gluing inside solid tori. Thus we can assume that all $n_i's$ are equal to 2. From Theorem 1.1 and Theorem 4.3 in [KM] we see that $c_{2,2,\ldots,2}$ and $J(L,2,2,\ldots,2)$ satisfy the same skein relation so the equality follows from that for trivial links.

Let $\mathbf{n} = (n_1, n_2, \cdots, n_k)$ be arbitrary. Define $\mu(\mathbf{n}) = max_i(n_i)$ and $\nu(\mathbf{n}) = card\{i|n_i = \mu(\mathbf{n})\}$. We will prove the theorem by induction on $(\mu(\mathbf{n}), \nu(\mathbf{n}))$, where the pairs are ordered lexicographically. We saw above that the result is true for arbitrary links if $\mu(\mathbf{n}) \leq \mathbf{2}$. Suppose that it is true for all links and all $\mathbf{n}'$ with $(\mu(\mathbf{n}'), \nu(\mathbf{n}')) < (\mu(\mathbf{n}), \nu(\mathbf{n}))$. Let us prove it for $(\mu(\mathbf{n}), \nu(\mathbf{n}))$. Suppose that the k'th boundary component has $n_k = \mu(\mathbf{n})$. Consider the mapping cylinder of the identity on a pair of pants and glue the ends together to get the ce-manifold described in Fig. 4.

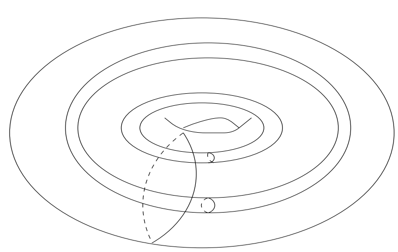

Figure 4.

It's invariant is $\delta_w^{uv}\beta_w^w \otimes \beta_u^u \otimes \beta_v^v$, where $\delta_w^{uv} = 1$ if $|u-v|+1 \leq w \leq min\{u+v-1, 2r-1-u-v\}$, and $u+v+w$ odd, and 0 otherwise.

If we glue this manifold along the exterior torus to the k'th boundary component of M, such that the meridian goes to the decomposition circle in M we get a manifold M'. Remark that this manifold is actually obtained by removing a regular neighborhood of the link L' that one gets from L by doubling the k'th component. Now choose $u, v < n_k$, such that $u+v = n_k + 1$. Then

$$\underline{u} \otimes \underline{v} = \underline{n_k} \oplus \underline{n_k - 2} \oplus \cdots.$$

Let $Z(M', D', 0) = 1/X \sum d_{\mathbf{m}} \beta_{\mathbf{m}}^{\mathbf{m}}$. If $\mathbf{m} = (n_1, n_2, \cdots, n_{k-1}, u, v)$, then by the induction hypothesis $d_{\mathbf{m}} = J(L', \mathbf{m})$. From the gluing axiom we have $Z(M', D', 0) = 1/X \sum_{\mathbf{m}} f(\mathbf{m}) \beta_{\mathbf{m}}^{\mathbf{m}}$ where $\mathbf{m} = (n_1, n_2, \cdots, n_{k-1}, u, v)$ and $f(\mathbf{m}) = \sum_w c_{n_1, \cdots, n_{k-1}, w} \delta_w^{uv}$.

On the other hand, from the induction hypothesis $c_{n_1, \cdots, n_{k-1}, w} = J(L', n_1, \cdots, n_{k-1}, w)$ for all $w < n_k$, hence

$$c_{n_1, \cdots, n_{k-1}, n_k} = J(L', n_1, \cdots, n_{k-1}, u, v) - \sum_{w < n_k} J(L, n_1, \cdots, n_{k-1}, w) \delta_w^{uv}.$$

But $c_{n_1, n_2 \cdots, n_k} = J(L, n_1, n_2 \cdots, n_k)$ satisfies this equality, and the proposition is proved.□

The following corollary will show that our invariants for closed manifolds are the Reshetikhin-Turaev invariants multiplied by X^{-1}. This will give an understanding of these invariants from the point of view of TQFT with corners.

Corollary 1.3. If M is the 3-manifold obtained by performing surgery on the framed link L with k components, then

$$Z(M, 0) = X^{-k-1} C^{-\sigma} \sum_{\mathbf{m}} [\mathbf{m}] J(L, \mathbf{m})$$

where σ is the signature of the linking matrix and $C = exp(3(r-2)\pi i/(4r))$.

Proof: The conclusion follows from Proposition 1.2 by gluing solid tori along the boundary components of the complement of the regular neighborhood of the link, and using the gluing axiom [Wa]. Using Wall's Theorem [W] we get that the framing of the new manifold is the signature of the link, so to get framing equal to 0 one has to apply the map $(id, -1)$ σ times, which gives the factor $C^{-\sigma}$ from the formula.□

2. INVARIANTS OF TORUS KNOTS

As an application we will show how TQFT with corners can be used to compute the colored Jones polynomials of torus knots. We will need the following lemma. Its proof is elementary, making use of the fact that the sum of the roots of unity is equal to 0, and is left to the reader.

Lemma 2.1. Let a, b, c, d and e be integers. Then

$$\sum_{x,y=1}^{r}[ax]t^{bx^2}[cy]([x(y+d)]t^{2ey}+[x(y-d)]t^{-2ey})=$$
$$X^2t^{bc^2+be^2-2de}([a(c+e)]t^{2(be-d)c}+[a(c-e)]t^{-2(be-d)c}).$$

Let $T_{(p,q)}$ be the (p,q)-torus knot. We construct the complement of a regular neighborhood of this knot by first considering a solid torus, then gluing along the (p,q)-curve on its boundary another solid torus with its core removed, and finally, filling the exterior with a third torus.

The invariant of the solid torus, viewed as a cylinder with ends glued together via the identity map is $(1/X)\beta_1^1$. In this case the decomposition curve is a meridian. To transform this meridian into the (p,q)-curve, consider the continuous fraction expansion

$$\frac{q}{p}=-\frac{1}{-a_1-\frac{1}{-a_2-\cdots\frac{1}{-a_n}}}.$$

The required move will be $ST^{-a_n}ST^{-a_{n1}}S\cdots ST^{-a_1}S$, where the moves S and T on the torus are described in Fig. 5. On the vector space S is the matrix $([mn])_{m,n=1}^{r-1}$, and T is the matrix $(\delta_{m,n}t^{-m^2+1})_{m,n=1}^{r-1}$.

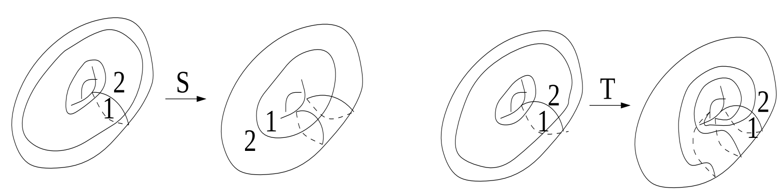

Figure 5.

The invariant of the torus in the new DAP-decomposition is

$$X^{-n-2}\sum_{j_1,\cdots,j_{n+1}}[j_{n+1}j_n]t^{a_n(j_n^2-1)}[j_nj_{n-1}]\cdots[j_2j_1]t^{a_1(j_1^2-1)}[j_1]\beta_{j_{n+1}}^{j_{n+1}}.$$

Expand an annulus along the (p,q)-curve. Take the complement of a regular neighborhood of the Hopf link with the canonical DAP-decomposition, expand an annulus in one of its boundary components and glue it along this annulus to the annulus that we expanded in the initial torus. The invariant of the complement of the Hopf link is $1/X\sum_{m,j_{n+1}}[mj_{n+1}]\beta_m^m\otimes\beta_{j_{n+1}}^{j_{n+1}}$, and the gluing along the annulus introduces a factor of $X/[j_{n+1}]$, thus the invariant of the new manifold, after contracting an annulus in its boundary, will be

$$X^{-n-2}\sum_{j_1,\cdots,j_{n+1},m}\frac{[mj_{n+1}]}{[j_{n+1}]}[j_{n+1}j_n]t^{a_n(j_n^2-1)}[j_nj_{n-1}]\cdots[j_2j_1]t^{a_1(j_1^2-1)}[j_1]\beta_{j_{n+1}}^{j_{n+1}}\otimes\beta_m^m.$$

Apply the move $(ST^{-a_n}ST^{-a_{n_1}}S\cdots ST^{-a_1}S)^{-1}$ on the exterior torus to get back its initial DAP-decomposition, then glue along it a solid torus so that we get the complement of the regular neighborhood of $T_{(p,q)}$. The invariant of the ce-manifold we have constructed is

$$X^{-2n-3}\sum_{j_1,\cdots,j_{2n+2},m}[j_{2n+2}][j_{2n+2}j_{2n+1}]t^{-a_1j_{2n+1}^2}\cdots[j_{n+2}j_{n+1}]\frac{[mj_{n+1}]}{[j_{n+1}]}[j_{n+1}j_n]t^{a_nj_n^2}[j_nj_{n-1}]\cdots[j_2j_1]$$
$$t^{a_1j_1^2}[j_1]\beta_m^m.$$

Note that the DAP-decomposition that we now have is not the canonical one since the seams on the boundary are twisted by a T^{-q}. To get the right DAP-decomposition we must multiply by $t^{-q(m^2-1)}$.

On the other hand

$$\frac{[mj_{n+1}]}{[j_{n+1}]}=\sum_{\substack{1\le k\le m-1\\ m-k\equiv 1(mod2)}}(t^{2mj_{n+1}}+t^{-2mj_{n+1}})$$

if m is even, and

$$\frac{[mj_{n+1}]}{[j_{n+1}]}=1+\sum_{\substack{1< k\le m-1\\ m-k\equiv 1(mod2)}}(t^{2mj_{n+1}}+t^{-2mj_{n+1}})$$

if m is odd.

Therefore, for m even, it follows from Proposition 1.2 that

$$J(T_{(p,q)},m)=X^{-2n-2}t^{-q(m^2-1)}\sum_{j_1,\cdots,j_{2n+2},m}[j_{2n+2}][j_{2n+2}j_{2n+1}]t^{-a_1j_{2n+1}^2}\cdots[j_{n+2}j_{n+1}]\times$$
$$\sum_{\substack{1\le k\le m-1\\ m-k\equiv 1(mod2)}}(t^{2mj_{n+1}}+t^{-2mj_{n+1}})[j_{n+1}j_n]t^{a_nj_n^2}[j_nj_{n-1}]\cdots[j_2j_1]t^{a_1(j_1^2)}[j_1].$$

A succesive application of Lemma 2.1 starting in the middle of the expression gives

$$J(T_{(p,q)},m)=X^{-2n}t^{-q(m^2-1)}\sum_{j_1,\cdots,j_{2n+2},m,k}t^{a_nk^2}[j_{2n+2}][j_{2n+2}j_{2n+1}]t^{-a_1j_{2n+1}^2}\cdots[j_{n+3}j_{n+2}]\times$$

$$([j_{n-1}(j_{n+2}+k)]t^{2a_nkj_n+2}+[j_{n-1}(j_{n+2}-k)]t^{-2a_nkj_n+2})t^{a_{n-1}j_{n-1}^2}[j_{n-1}j_{n-2}]\cdots[j_2j_1]t^{a_1j_1^2}[j_1]=$$

$$X^{-2n+2}t^{-q(m^2-1)}\sum_{j_1,\cdots,j_{2n+2},m,k}t^{a_n(a_na_{n-1}-1)k^2}[j_{2n+2}][j_{2n+2}j_{2n+1}]t^{-a_1j_{2n+1}^2}\cdots[j_{n+4}j_{n+3}]\times$$

$$([j_{n-2}(j_{n+3}+a_nk)]t^{2(a_na_{n-1}-1)kj_n+3}+[j_{n-2}(j_{n+3}-a_nk)]t^{-2(a_na_{n-1}-1)kj_n+3})t^{a_{n-2}j_{n-2}^2}\times$$

$$[j_{n-2}j_{n-3}]\cdots[j_2j_1]t^{a_1j_1^2}[j_1]=\cdots=X^{-2}t^{-q(m^2-1)}\sum_{j_{n+2},j_{n+1},k}t^{pqk^2}[j_{2n+2}][j_{2n+2}j_{2n+1}]([j_{2n+1}+kq]\times$$

$$t^{2kpj_{2n+1}}+[j_{2n+1}-kq]t^{-2kpj_{2n+1}})=t^{-q(m^2-1)}\sum_{\substack{1\le k\le m-1\\ m-k\equiv 1(mod 2)}}t^{pqk^2}([1+kq]t^{2kp}+[1-kq]t^{-2kp}).$$

A similar computation can be done for m odd. The formulas from both cases can be expressed in the more compact form

$$J(T_{(p,q)},m)=t^{-q(m^2-1)}\sum_{k=0}^{m-1}t^{pq(2k-m+1)^2+2p(2k-m+1)}[q(2k-m+1)+1]$$

Remarks: 1. In the case when $q=2$ one gets the formula obtained in [La].

2. The same argument shows that if $K_{p,q}$ is the (p,q)-cabling of the knot K, then

$$J(K_{p,q},m)=t^{-q(m^2-1)}\sum_{k=0}^{m_1}t^{pq(2k-m+1)^2+2p(2k-m+1)}J(K,q(2k-m+1)+1).$$

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