$SL(2, C)$-TOPOLOGICAL QUANTUM FIELD THEORY WITH CORNERS

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ABSTRACT

In this paper we define the $sl(2, \mathbf{C})$ topological quantum field theory with *corners* that corresponds to the smooth theory of Reshetikhin and Turaev. We encounter a sign obstruction at the level of the modular functor, which we solve by making use of the Klein four group. We deduce the Moore-Seiberg equations in the new context.

Keywords: topological quantum field theory, 3-manifolds

0. Introduction

This paper contains the basic ideas for the construction of a topological quantum field theory (TQFT) with corners associated to the Reshetikhin-Turaev quantum invariants. This construction is done via irreducible representations of quantum deformations of $sl(2, \mathbf{C})$ and gives rise to the smooth TQFT of Reshetikhin and Turaev [RT]. Another TQFT with corners was defined in [G1], this one underlying the smooth TQFT of Blanchet, Habegger, Masbaum and Vogel [BHMV]. The two theories are related by the fact that they give rise to the same invariants for closed manifolds, however, they are fundamentally distinct, in the sense that the previous one is naturally related to the Kauffman bracket, while the present one corresponds to the Jones polynomial.

The first example of a TQFT with corners belongs to Turaev and Viro [TV]. They defined invariants of three-manifolds by making use of triangulations. The invariants obtained this way were less fine than the Reshetikhin-Turaev invariants [Wa].

For the construction of a TQFT with corners corresponding to the Reshetikh–in-Turaev invariants, Walker [Wa] had the idea of using handle decompositions of manifolds. In his approach he introduces an initial amount of information (basic data), and a set of rules (axioms) that enable the computation of all invariants out of those for simple manifolds. The basic data is subject to satisfying the Moore-Seiberg equations [MS]. Walker's work was left unfinished and was continued by Frohman and Kania-Bartoszynska [FK].

When defining the basic data, the three above mentioned authors encountered a sign obstruction at the level of the Moore-Seiberg equations. For this reason in [FK] the authors use only the odd-dimensional representations, and the theory they construct no longer underlies the Reshetikhin-Turaev smooth theory. As explained in [G2], this sign obstruction is naturally related to the fact that the skein relation of the Jones polynomial is defined for oriented links. In the present paper we explain how the sign problem can be solved by using an additional structure on the boundary of three-manifolds. The structure that we add consists of a collection of simple closed curves indexed by the elements of the Klein four group. Our approach is similar to that of adding a Lagrangian space ([Wa], [T]) in order to solve the projective ambiguity of the invariants, and fits in the formalism of Chap. III in [T]. We deduce the Moore-Seiberg equations in the new context and adapt the techniques from [FK] to obtain the $sl(2,\mathbf{C})$-topological quantum field theory.

The author thanks Ch. Frohman and N. Reshetikhin for the fruitful discussions.

1. Completely Extended Surfaces and 3-Manifolds

In this section we introduce, on the boundary of a 3-manifold, the Klein four group structure that will make the modular functor well defined. Before we proceed with our definitions, let us explain how the Klein group appears in this context. For this discussion we assume that the reader is familiar with the formalism from [RT].

Consider a strand in the plane that looks like the graph of a function with several local maxima and minima on an interval, with a minimum at the beginning of the interval, and a maximum at the end of the interval. Label this strand by an irreducible representation V and fix a direction on it, such that the graph will represent a morphism from V into V. Let $K = \{e, a, b, c\}$ be the Klein four group, where we recall that $a^2 = b^2 = c^2 = e$, $ab = c$, $bc = a$, $ac = b$. K acts on the space of morphisms from V to V in the following way: multiplication by a inserts at a point where the graph is increasing a coupon corresponding to the isomorphism w_i between V and its dual, and multiplication by b inserts such a coupon at the point where the graph is decreasing (see Fig. 1.1). It is not hard to see that when we insert a or b twice we get back what we started with [KM], hence the action is well defined. In this section we will find the analogue of this action on the boundary of 3-manifolds.

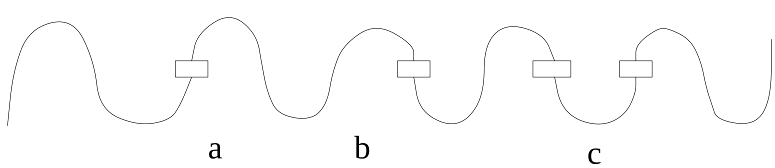

Fig. 1.1.

Let Σ be a compact, orientable piecewise linear (non-necessarily closed) surface. As defined in [Wa], a DAP-decomposition of Σ will consist out of a collection of disjoint curves in the interior of Σ , which cut the surface into an ordered collection of basic surfaces (disks, annuli and pairs of pants); a numbering of the boundary components of the basic surfaces, by 1 for disks, 1 and 2 for annuli, and 1, 2 and 3 for pairs of pants; a parametrization of the boundary components of basic surfaces by S^1, such that the parametrization two parametrizations coming from neighboring surfaces should be one the complex conjugate of the other; and finally, on each basic surface, disjoint arcs joining the point $e^{i\epsilon}$ on the the j-th boundary component with $e^{-i\epsilon}$ on the (j+1)-st boundary component.

Definition: A completely extended surface is a triple (Σ, D, B) where

(i) Σ is a compact, oriented, piecewise linear surface,

(ii) D is a DAP-decomposition,

(iii) B is a collection of bands (simple closed curve) such that each basic surface Σ_0 from the DAP-decomposition of Σ contains one band parallel to each of its boundary. The bands are indexed by e, a, b, or c according to the following rules: if the two numbers that a decomposition circle has in two neighboring basic surfaces are both 1, or none of them is 1, then the product of the indices of the two bands parallel to that component should have the product equal to a or b; in the other cases the product of those indices should be e or c.

The reason why we have introduced the bands instead of simply indexing the decomposition curves is that we want to make gluings with corners possible in all situations. For simplicity we will make the notations described in Fig 1.2.

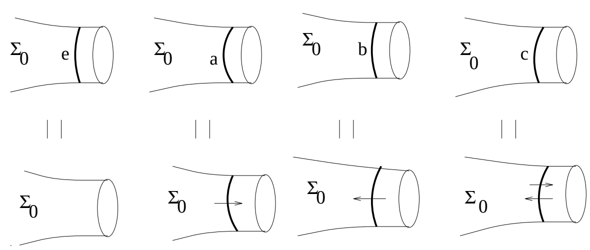

Fig. 1.2.

We will factor the set of completely extended surfaces (shortly ce-surfaces) by the following two identifications:

1. We multiply the indices of two bands parallel to the same decomposition circle by u and respectively cu for $u = a$ or b, or we multiply them both with c. In other words we allow "arrows" to slide over decomposition disks.

2. If $(\Sigma', D', B') \subset (\Sigma, D, B)$ is a subsurface, let us consider the ce-surface obtained from (Σ, D, B) by multiplying all bands along the boundary components of Σ' by c. Identify the newly obtained ce-surface with (Σ, D, B). This means that we are allowed to reverse one "arrow" on each band along the boundary of a subsurface of Σ .

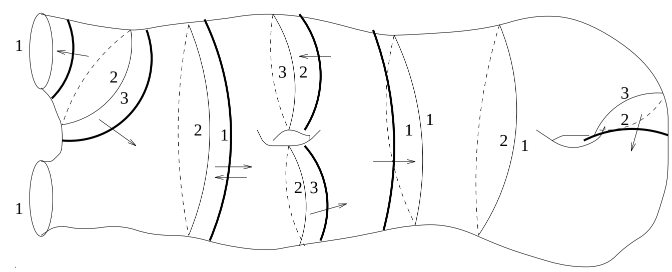

Fig. 1.3.

An example of an extended surface is given in Fig. 1.3. From now on the structure that makes a surface into a ce-surface will be called DB-structure. The transformations between DB-structures will be called moves. At the level of the modular functor, they correspond to changes of basis in the associated vector space. All moves can be obtained by composing the elementary moves described in Fig. 1.4. Note that the F and S moves can be applied only when the arrows point in the right direction. For the general case, one has to compose them with a K-move. Note also that the S-move on the torus was not described above, however, it is not hard to write it as a composition of elementary moves.

We say that a boundary component of a surface is of positive type if it is numbered by 1 and its associated band is indexed by e or c, or if it is numbered by 2 or 3 and its associated band is indexed by a or b. In all other cases we say that the boundary component is of negative type. We allow two surfaces to be glued only along boundary components that have opposite type.

The dual of a ce-surface is defined by $-(\Sigma, D, B) := (-\Sigma, -D, -B)$ where $-\Sigma :=\Sigma$ with opposite orientation, $-D$ is obtained from D by changing the orientation of each disk, annulus or pair of pants from the decomposition of Σ and $-B$ is defined as follows: inside Σ the band structure remains unchanged; we multiply by a the index of all bands along boundary components of positive

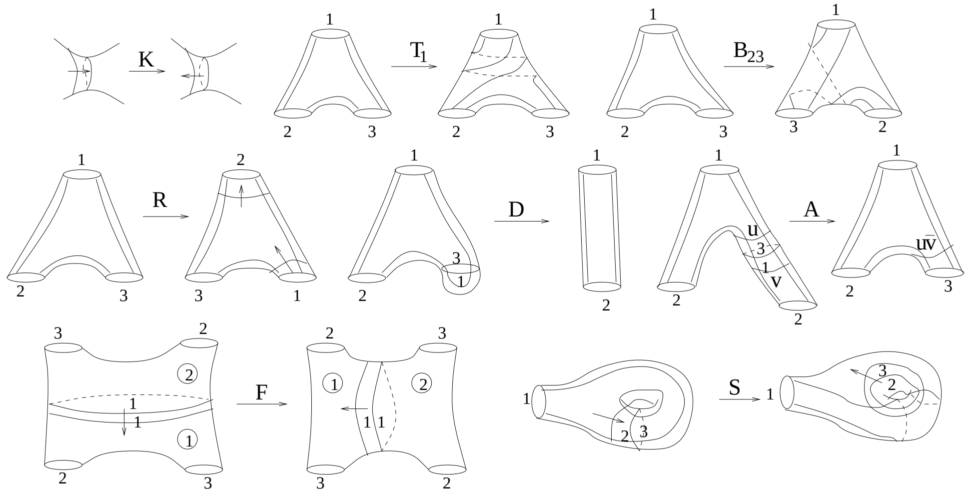

Fig. 1.4.

type, and by b the index for boundary components of negative type. Note that by the identifications we made for DB-structures the operation of taking the dual is reflexive.

A ce-morphism is a map between two ce-surfaces $(f, n) : (\Sigma_1, D_1, B_1) \to (\Sigma_2, D_2, B_2)$ where f is a homeomorphism satisfying the property that for any boundary component C of Σ_1, C and $f(C)$ are of the same type, and n is an integer. From this definition it follows that the mapping class groupoid of the ce-surfaces having the same underlying surface has exactly as many connected components as the number of the boundary components of the surface. However, this produces no restrictions on the gluings along surfaces with boundary, since by expanding an annulus via a move of type A and sliding an "arrow" into that annulus we can always adjust the surface such that the gluing can be done.

If $(f_1, n_1) : (\Sigma_1, D_1, B_1) \to (\Sigma_2, D_2, B_2)$ and $(f_2, n_2) : (\Sigma_2, D_2, B_2) \to (\Sigma_3, D_3, B_3)$ are ce-morphisms, following [Wa], we define their composition

$$(f_2, n_2)(f_1, n_1) := (f_2 f_1, n_2 + n_1 - \sigma((f_2 f_1)_* L_1, (f_2)_* L_2, L_3))$$

where L_i is the subspace of $H_1(\Sigma_i)$ generated by D_i (including the curves on the boundary), $i = 1, 2, 3$ and σ is Wall's non-additivity function ([W]).

Definition: We say that (M, D, B, n) is a completely extended 3-manifold (ce-3-manifold) if M is a compact, oriented, piecewise linear 3-manifold, D and B determine a DB structure on ∂M, and n is an integer (usually called framing).

The definition of the mapping cylinder, disjoint union, and gluings for ce-manifolds are identical to those defined by Walker [Wa] for e-manifolds, with the only remark that the gluing is done via ce-morphisms, and the Klein group structure on the boundary is the one obtained by keeping the indices of the bands that lie in the complements of the two surfaces that we glue along. For the correct definition of how the framing changes, see [G1].

2. The Basic Data for the Modular Functor

In this section we are going to describe a modular functor V from the category of labeled ce-surfaces to the category of finite dimensional vector spaces. The partition function Z associated to it can be constructed from V via Walker's axioms for a TQFT with corners (see [Wa] or [G1]). As shown in [Wa], in order to define the modular functor one only needs to specify the basic data. This is what we will do in this section. What we need is an assignation of vector spaces to labeled, completely extended disks, annuli and pairs of pants, a choice of basis elements in these vector spaces, a duality pairing for these spaces, a definition of the linear morphisms that correspond to the moves K_i, T_i, R, B_{23}, F, S, P, A, D, $C=(id,1)$ and a function $S:\mathcal{L}\to\mathbf{C}^*$ (needed for the pairing (see [Wa]).

For the construction we will use the representation theory of $U_t(sl(2,\mathbf{C})$ and the diagram formalism associated to it, that were discussed extensively in [RT] and [KM]. We will denote the Weyl element, (i.e. the morphism between a representation and its dual) by an empty coupon.

Let r be the level of the TQFT. The label set $\mathcal{L}$ consists of the irreducible representations V_n , $n=1,2,\cdots,r-1$ of $U_t(sl(2,\mathbf{C})$ considered in [RT].

If D is a ce-disk, define $V(D,m)=V_m$ where V_m is the space of module homomorphisms $\mathbf{C}\to\underline{m}$. By Schur's lemma $V_m=0$ unless $m=1$, and

$V_1 \tilde{=} \mathbf{C}$. Let $\beta_1 : \mathbf{C} \to \underline{1}$ be the isomorphism $\lambda \to \lambda e_0$. If A is a ce-annulus, $V(A,(m,n)) = V_n^m = \{\phi : \underline{m} \to \underline{n}\ \phi$ module homomorphism$\}$. Like before, $V(A,(m,n)) \neq 0$ if and only if $\underline{m} = \underline{n}$. In the latter case $V(A,(m,n)) \tilde{=} \mathbf{C}$. Let β_m^m be the identity operator on $\underline{m}$.

If P is a ce-pair of pants define $V(P,(p,m,n)) = V_p^{mn}$. It follows that $V(P,(p,m,n)) \neq 0$ if and only if $|m-n|+1 \leq p \leq min\{m+n-1, 2r-2-m-n\}$, in which case the vector space is isomorphic to $\mathbf{C}$. An observation that is very useful in computations is the fact that $V_p^{mn} \neq 0$ implies that $m+n+p$ is an odd integer, and $m-1+n-1+p-1$ is even. Denote by β_p^{mn} one of the two projections in V_p^{mn}. One can choose these projections such that $R\beta_p^{mn} = \beta_n^{pm}$ where R will be defined below. In a diagram, the morphism β_p^{mn} will be denoted like in Fig. 2.1.a). Choose β_1^{mm} to be equal to the diagram from Fig. 2.1.b).

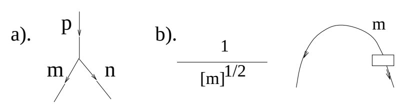

Fig. 2.1.

In order to be able to perform gluings, we need to specify the duals of these spaces. The dual of V_1 is also V_1, and the pairing is given by $<\beta_1, \beta_1> = 1$, the dual of V_m^m is V_m^m, with pairing given by $<\beta_m^m, \beta_m^m> = X/[m]$, and the dual of V_p^{mn} can be identified with V_p^{nm}, with pairing given by $<\beta_p^{mn}, \beta_p^{mn}> = X^2/(\sqrt{[m]}\sqrt{[n]}\sqrt{[p]})$

The following lemma will be needed in the next section.

Lemma 2.1. If $\alpha \in V_p^{mn}$ and $\beta \in V_p^{nm}$ then $<\alpha, \beta> = <\beta, \alpha>$.

Proof: Since the space V_p^{nm} is one dimensional we can assume that $\beta = c \cdot \alpha \circ \check{R}$, where c is some complex number, thus $<\alpha, \beta> = c <\alpha, \alpha \circ \check{R}>$. Using the equality from Fig. 4.5 we get that this is equal to $= c <\alpha \circ \check{R}, \alpha>$.

We now describe the linear morphisms associated to elementary moves. They will be described as diagrams. For simplicity, we will denote $V(f)$ also by f. It might come as a surprise the fact that we used diagrams that are mirror images of the diagrams that usually appear in literature. However, *this* is the

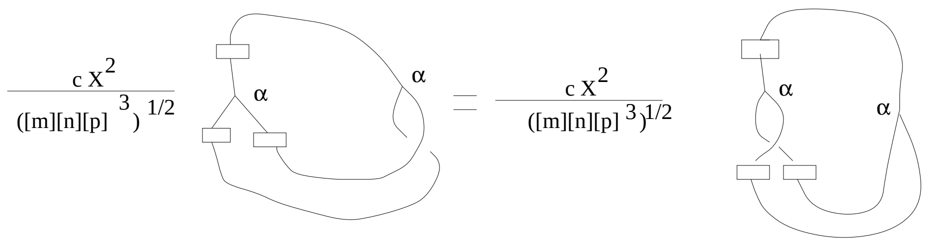

Fig. 2.2.

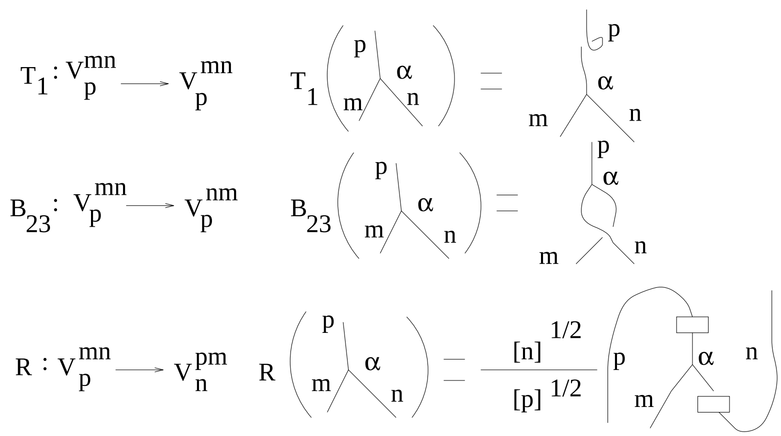

Fig 2.3.

modular functor that produces the Reshetikhin-Turaev invariants, if we used the other diagrams, we would be computing the invariants for the manifolds with opposite orientation. Our choice is due to the fact that moves correspond to changes of bases in the vector space, and usually the relationship between an isomorphism and a change of basis is that if the isomorphism has the matrix A, then the change of basis which transforms A into the identity has the matrix equal to A^{-1}.

To the K-move applied to a band labeled by m we associate the operator of multiplication by $(-1)^{m-1}$. For an lce-pair of pants with boundary components labeled respectively p, m, n define T_1, B_{23} and R like in Fig. 2.3.

For two pairs of pants glued along a boundary component we let

$$F : \bigoplus_{p\in\mathcal{L}} V_p^{mn} \otimes V_p^{kl} \to \bigotimes_{q\in\mathcal{L}} V_q^{lm} \otimes V_q^{nk}$$

be defined by $F(\beta_p^{mn} \otimes \beta_p^{kl}\ ) = \sum_q f_{mnklpq}\beta_q^{lm} \otimes \beta_q^{nk}$ where f_{mnklpq} is the value of the diagram in Fig. 2.4.a) (which is a multiple of the quantum 6j-symbol).

For an lce-pair of pants, with the bottom boundary components glued together we define $S : \bigoplus_{m\in\mathcal{L}} V_p^{mm} \to \bigoplus_{n\in\mathcal{L}} V_p^{nn}$, by the diagram in Fig. 2.4.b).

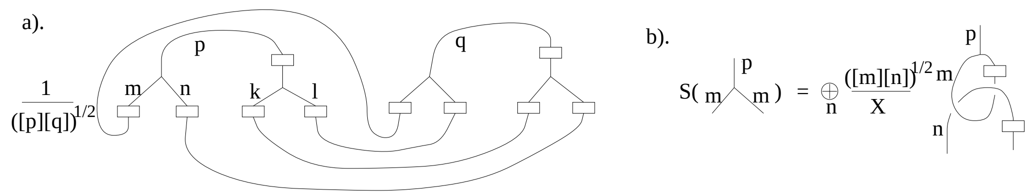

Fig. 2.4.

Also $P(\alpha\otimes\beta) = (\beta\otimes\alpha)$, $A(\beta_p^{mn}\otimes\beta_m^m) = \beta_p^{mn}$, $D(\beta_m^{1m}\otimes\beta_1) = \beta_m^m$, $A(\beta_m^m\otimes\beta_m^m) = \beta_m^m$, $D(\beta_1^1\otimes\beta_1) = \beta_1$. The map $C = V(id, 1) : V(\sigma) \to V(\sigma)$ is multiplication by $exp(3\pi(r-2)i/(4r))$, and finally $S : \mathcal{L} \to \mathbf{C}^*$ is given by $S(m) = [m]/X$.

3. The compatibility conditions

In order for the TQFT to be well defined, the basic data must satisfy certain compatibility conditions. The first six groups of conditions are the analogues of the Moore-Seiberg equations for the case of completely extended surfaces. The last two relations must hold for the partition function to be well defined (see [Wa]).

Theorem 3.1. A basic data determines a well defined modular functor V and a partition function Z if and only if the following relations hold:

1. relations at the level of a ce-pair of pants:

a) $T_1B_{23} = B_{23}T_1$, $T_2B_{23} = B_{23}T_3$, $T_3B_{23} = B_{23}T_2$, where $T_2 = RT_1R^{-1}$ and $T_3 = R^{-1}T_1R$.

b) $B_{23}^2 = T_1T_2^{-1}T_3^{-1}$

c) $R^3 = 1$

d) $RB_{23}R^2B_{23}RB_{23}R^2 = B_{23}RB_{23}R^2B_{23}$

e) $K^2 = 1$

2. relations defining the inverses of F and S:

a) $P_{12}K_1^{(1)}F^2 = 1$

b) $K_2T_3^{-1}B_{23}^{-1}S^2 = 1$

3. relations coming from triangle singularities:

a) $K_1^{(1)}P_{13}R^{(2)}F^{(12)}K_1^{(1)}R^{(2)}K_1^{(2)}F^{(23)}R^{(2)}F^{(12)}K_1^{(1)}R^{(2)}K_1^{(2)}F^{(23)}R^{(2)}F^{(12)} = 1$

b) $T_3^{(1)}FB_{23}^{(1)}FB_{23}^{(1)}FB_{23}^{(1)} = 1$

c) $C^{-1}K_2B_{23}^{-1}T_3^{-2}ST_3^{-1}ST_3^{-1}S = 1$

d) $R^{(1)}(R^{(2)})^{-1}FS^{(1)}FB_{23}^{(2)}B_{23}^{(1)} = K_1^{(1)}FS^{(2)}T_3^{(2)}(T_1^{(2)})^{-1}B_{23}^{(2)}F$

4. relations involving annuli and disks:

a) $F(\beta_p^{mn}\otimes\beta_p^{p1}) = \beta_m^{1m}\otimes\beta_m^{np}$

b) $A_2^{(12)}D_3^{(13)} = D_2D_3^{(13)}$

c) $A^{(12)}A^{(23)} = A^{(23)}A^{(12)}$

5. relations coming from duality:

for any elementary move f, one has $f^+ = \bar{f}$ where f^+ is the dual of f with respect to the $<\,,\,>$ pairing, and $\bar{f}$ is the morphism induced by f on $-\sigma$,

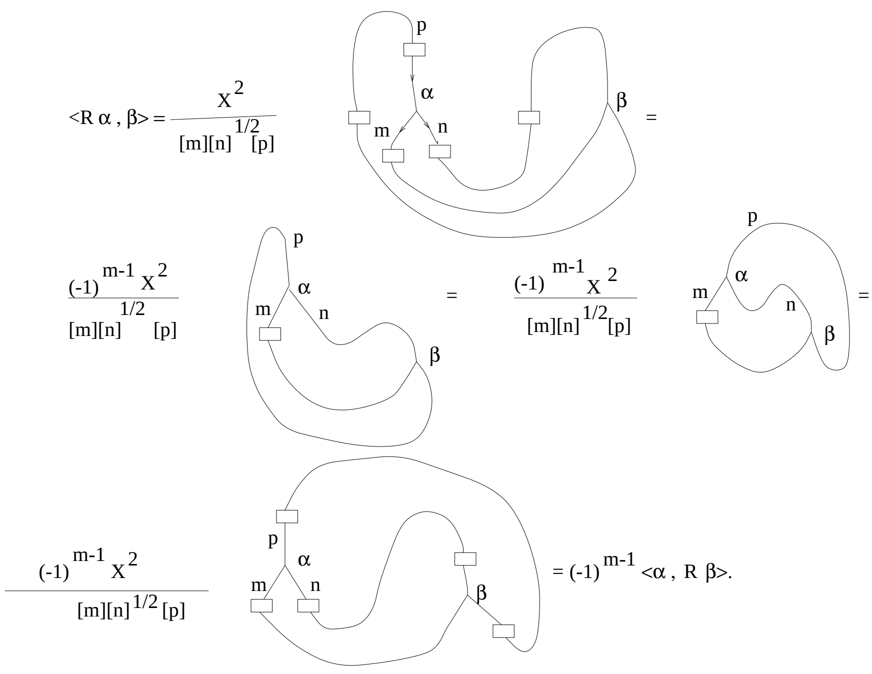

Fig. 3.2.

6. relations expressing the compatibility between the pairing, and moves A and D:

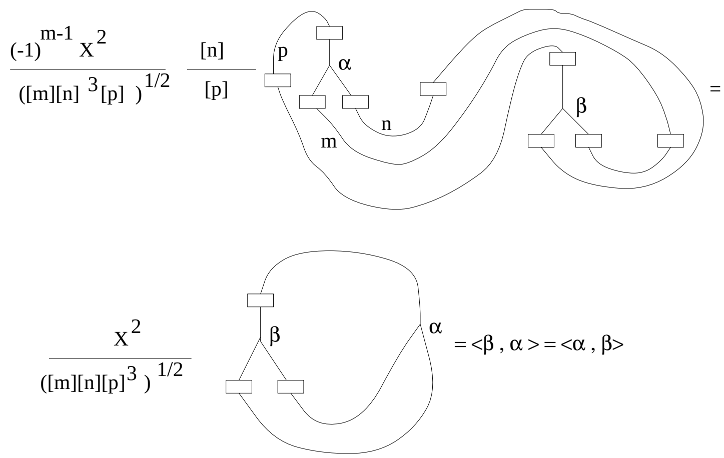

Fig. 3.3.

a) $< \beta_m^m, \beta_m^m >= S(m)^{-1}$

b) $< \beta_m^{m1}, \beta_m^{m1} >= S(1)^{-1} S(m)^{-1}$.

7. relations related to the partition function:

a) $S(m) = S_{1m}$, where $[S_{xy}]_{x,y}$ is the matrix of move S on the torus,

b) $F(\beta_1^{mm} \otimes \beta_1^{nn}) = \sum_p (-1)^{n-1} id_{mnp}/S(m)S(n)$, where id_{mnp} is the identity matrix in $(V_p^{mn})^* \otimes V_p^{mn}$.

In the above relations the superscripts indicate the elementary surfaces to which the move is applied, while the subscripts indicate the number of the boundary component. As the reader can see, there are no listed relations that are to be satisfied at the level of a torus. This is because those relations are redundant, the torus can be obtained by capping the punctured torus with a disk. This theorem can be proved like the analogous result from [Wa].

The fact that the basic data we defined in Section 2. satisfies the compatibility conditions can be checked like in [FK] or [G1]. We will check below only two of the compatibility conditions, which are usually omitted in the literature. The first one is $R^3 = 1$. Its proof for arbitrary quantum groups is extremely long [DJN], however in our situation there is a much shorter proof.

Lemma 3.1. If $\alpha \in V_p^{mn}$ and $\beta \in V_p^{nm}$ then

$$< R\alpha, \beta >= (-1)^{m-1} < \alpha, R\beta > .$$

Proof: The proof is described in Fig. 3.2.

Let us show that $R^3 = 1$, which is equivalent to showing that $< \alpha, R^3\beta >= (-1)^{m-1} < \alpha, \beta >$ for all α and β. Applying the lemma we get that for $\alpha \in V_p^{mn}$ and $\beta \in V_p^{nm}$, $< \alpha, R^3\beta >= (-1)^{m-1} < R\alpha, R^2\beta >$. From here the proof proceeds like in Fig. 3.3. where the last equality follows from Lemma 2.1.

The second relation we want to verify is 7. b). Note first that the identity matrix on V_p^{mn} has the form $\bigoplus_{m,n,p} \hat{\beta_p^{mn}} \otimes \beta_p^{mn}$, where $\hat{\beta_p^{mn}} \in V_p^{nm}$ is the element dual to β_p^{mn} with respect to the pairing. Hence we must show that

$$< F\beta_1^{mm} \otimes \beta_1^{nn}, \beta_p^{mn} \otimes \beta_p^{nm} >= \frac{(-1)^{n-1} < \beta_p^{mn}, \beta_p^{nm} >}{S(m)S(n)}.$$

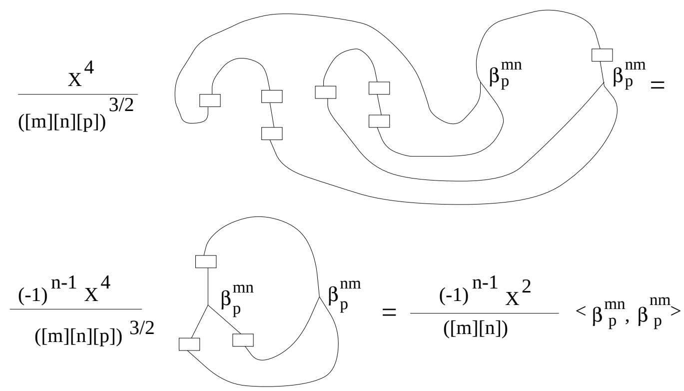

Fig. 3.4.

Using the definition of β_1^{mm} given in the previous section, we can transform the left hand side of this equality like in Fig. 3.4.

We conclude the paper by pointing out that the fact that this theory with corners produces the Reshetikhin-Turaev invariants for closed manifolds was shown in [G2].

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