Introduction to Topology – Homework 1

1. Check that the examples 2, 5, 6 from 1.3 describe indeed topologies.

2. Show that, given two topologies $\mathcal{T}$ and $\mathcal{T}'$ on the set X, the topology $\mathcal{T}$ is finer than $\mathcal{T}'$ if and only if the map $1_X : (X, \mathcal{T}) \to (X, \mathcal{T}')$, $1_X(x) = x$ for all x is continuous.

3. Show that the family
$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{Q}\}$$
is a basis for the standard topology on the real axis.

4. Show that the upper limit topology and the lower limit topology are different. Show that the collection
$$\mathcal{B} = \{[a, b) \mid a < b, a, b \in \mathbb{Q}\}$$
is a basis for a topology that is different from the lower limit topology. Show that this topology is coarser than the lower limit topology.

5. Show that if the topological space Y is a subspace of the topological space X, and A is a subset of Y, then the topology that A inherits from Y is the same as the topology that it inherits from X.

6. Show that the boundary of the square and the circle are homeomorphic.

7. Define $D : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by
$$D(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|.$$
Prove that D is a metric and that it induces the standard topology on $\mathbb{R}^n$.

8. Consider the 2-dimensional torus $S^1 \times S^1$ as a subspace of $\mathbb{C}^2 = \mathbb{R}^4$. Show that the subspace topology is the same as the quotient topology defined in Example 2 from §1.4.7.

9. Let X be a topological space. The suspension ΣX is defined as the quotient $X \times [-1, 1]/\sim$, where the equivalence relation is the following

- for $\lambda \neq -1, 1$, $(x, \lambda) \sim (y, \mu)$ if and only if $x = y$, $\lambda = \mu$;
- $(x, 1) \sim (y, 1)$ for all x, y;
- $(x, -1) \sim (y, -1)$ for all x, y.

Show that S^{n+1} is homeomorphic to ΣS^n, where S^n is the n-dimensional sphere.

10. Show that the sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$$

is a 2-dimensional manifold. (Hint. Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ be the North and the South poles. Take the stereographic projections onto the equatorial plane $\pi_N : S^2 \backslash \{N\} \to \mathbb{R}^2$ and $\pi_S : S^2 \backslash \{S\} \to \mathbb{R}^2$, then use the maps $f_N = \pi_N^{-1} : \mathbb{R}^2 \to S^2$ and $f_S = \pi_S^{-1} : \mathbb{R}^2 \to S^2$.)

11. Show that, as a real 2-dimensional manifold, $\mathbb{C}P^1$ is homeomorphic to the sphere S^2.