Introduction to Topology – Homework 2

1. Show that both open and closed balls in $\mathbb{R}^n$ are connected.
2. Show that the open unit ball in $\mathbb{R}^n$, $n \geq 2$, from which we remove the center is connected.
3. Show that the circle S^1 is path connected.
4. Show that a 1-dimensional and a 2-dimensional manifold are never homeomorphic.
5. Among all triangles of perimeter 1, does there exist one of maximal area? Which one?
6. Prove that $\mathbb{R}^2$ and S^2 are not homeomorphic.
7. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be distinct points in $\mathbb{R}^n$. Show that $\mathbb{R}^n \backslash \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is path connected.
8. Show that $\mathbb{R}^2 \backslash \mathbb{Q}^2$ is connected.
9. Show that the n-dimensional real projective space is compact.
10. Let A and B be compact subspaces of X and Y, respectively. Let also N be an open set in $X \times Y$ containing $A \times B$. Prove that there exist open sets U and V in X and Y, respectively, so that $A \times B \subset U \times V \subset N$.
11. Is the group $O(n, \mathbb{R})$ of orthogonal $n \times n$ matrices with real entries a compact subset of the set of $n \times n$ matrices? Here the space of $n \times n$ matrices is identified with $\mathbb{R}^{n^2}$ and is endowed with the standard topology?