Introduction to Topology – Homework 3

1. For two loops $f, g : S^1 \to S^1$, $f(0) = g(0) = f(1) = g(1)$ define the operation $f * g$ by $(f * g)(z) = f(z)g(z)$. Show that this operation induces a well defined operation on the set of equivalence classes of loops based at 1 modulo homotopy relative to the endpoint. Prove that this operation on the set of such equivalence classes is the same as the product of (equivalence classes of) loops.

2. Prove that the map $f : S^1 \to S^1$, $f(z) = z^3$ is a covering map.

3. Let G be a topological group (i.e. a group that is also a topological space such that $G \times G \to G$, $(g, g') \mapsto gg'$ and $G \to G$, $g \mapsto g^{-1}$ are continuous maps) that is path connected and locally path connected and let e be its identity. Let also $p : (\tilde{G}, \tilde{e}) \to (G, e)$ be a covering map. Show that there is a unique group structure on $\tilde{G}$ such that p is a group homomorphism.

4. Prove that the Möbius band (defined as $[0, 1] \times [0, 1]/ \sim$ where $(t, 0) \sim (1 - t, 1)$ for all $t \in [0, 1]$ and no other points are identified) is not simply connected.

5. Prove that if X, Y are path connected topological spaces, then

$$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y).$$

   Use this to find the fundamental groups of $S^1 \times S^1$, $S^1 \times S^2$, $S^1 \times \mathbb{R}$, $\mathbb{C} \setminus \{0\}$.