The Simplicial Homology of the Punctured Torus

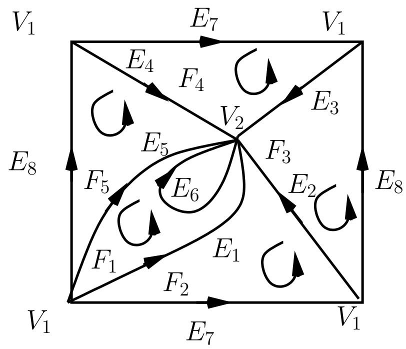

Figure 1:

$$0 \to \bigoplus_{j=1}^{5} \mathbb{Z}_j \to \bigoplus_{j=1}^{8} \mathbb{Z}E_j \to \mathbb{Z}V_1 \oplus \mathbb{Z}V_2 \to 0.$$

The boundary maps are

$$\partial_2 F_1 = E_1 + E_6 - E_5, \quad \partial_2 F_2 = E_7 + E_2 - E_1, \quad \partial_2 F_3 = E_8 + E_3 - E_2,$$
$$\partial_2 F_4 = -E_7 + E_4 - E_3, \quad \partial_2 F_5 = E_5 - E_8 - E_4,$$

and

$$\partial_1 E_1 = \partial_1 E_2 = \partial_1 E_3 = \partial_1 E_4 = \partial_1 E_5 = E_2 - E_1,$$
$$\partial_1 E_6 = \partial_1 E_7 = \partial_1 E_8 = 0.$$

Their matrices are

$$A_2 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

First, let us find the kernel of ∂_2, reading it off the matrix. If

$$\partial_2(n_1F_1 + n_2F_2 + n_3F_3 + n_4F_4 + n_5F_5) = 0,$$

then from the sixth row we see that $n_1 = 0$. Then the first row gives $n_2 = 0$, the second row gives $n_3 = 0$, the third $n_4 = 0$, and the fourth $n_5 = 0$. So $Z_2(X;\mathbb{Z}) = 0$, and hence $H_2(X;\mathbb{Z}) = 0$.

The kernel of ∂_1 has the basis

$$w_1 = E_1 - E_2, w_2 = E_1 - E_3, w_3 = E_1 - E_4, w_4 = E_1 - E_5, w_5 = E_6, w_6 = E_7, w_7 = E_8.$$

The image of ∂_2 is the column space of ∂_2. If we add all columns to the first, we obtain the matrix

$$A_2' = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

and after moving the first column to the end we realize that the following vectors form a basis:

$$E_1 - E_2 + E_7 = w_1 + w_6, E_2 - E_3 + E_8 = w_1 - w_2 + w_7, E_3 - E_4 - E_7 = w_2 - w_3 + w_6,$$
$$E_1 - E_5 - E_8 = w_3 - w_4 + w_7, w_5 = E_6.$$

We can arrange these as rows in a matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The row echelon form of this is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus we factor $\bigoplus_{j=1}^{7} \mathbb{Z}w_j$ by $w_1 + w_6, w_2 + w_6 - w_7$, $w_3 - w_7$, $w_4 - 2w_7$, w_6, and we obtain

$$H_1(X; \mathbb{Z}) = \mathbb{Z}^2.$$

Finally,

$$H_0(X; \mathbb{Z}) = (\mathbb{Z}V_1 \oplus \mathbb{Z}V_2)/\mathbb{Z}(V_1 - V_2) = \mathbb{Z}.$$