Math 3310
INDUCTION
Notes by Razvan Gelca

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Here are some examples:

1. Prove that for for every positive integer n*,*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

In other words, the sum of the first n *positive integers is equal to* $n(n+1)/2$*.*

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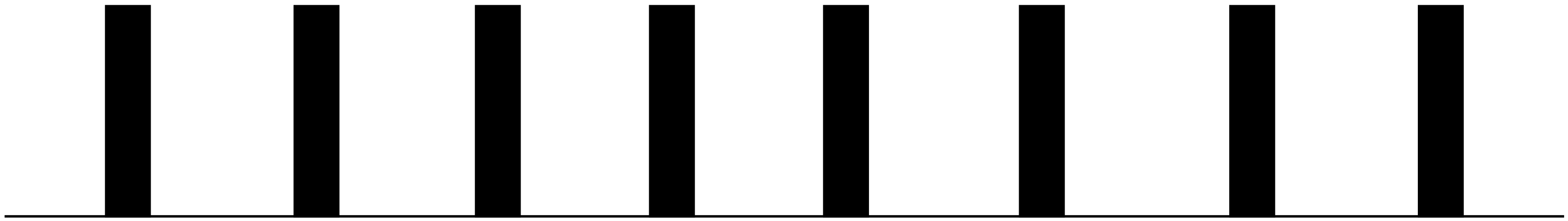
2. *Prove that for every positive integer* n*, the number* $n^{11}-n$ *is divisible by* 11*.*

3. *For a positive integer* n *you are given* n *lines divide the plane into regions. Show that the plane can be colored by two colors such that neighboring regions have different colors.*

Proofs by induction resemble dominoes falling



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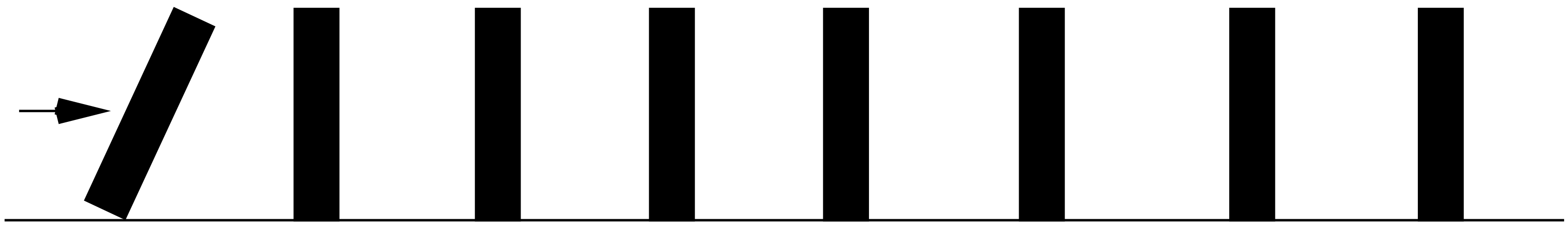

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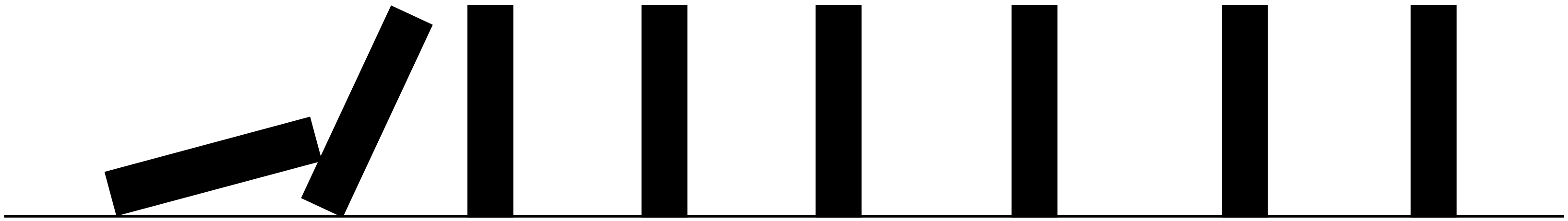

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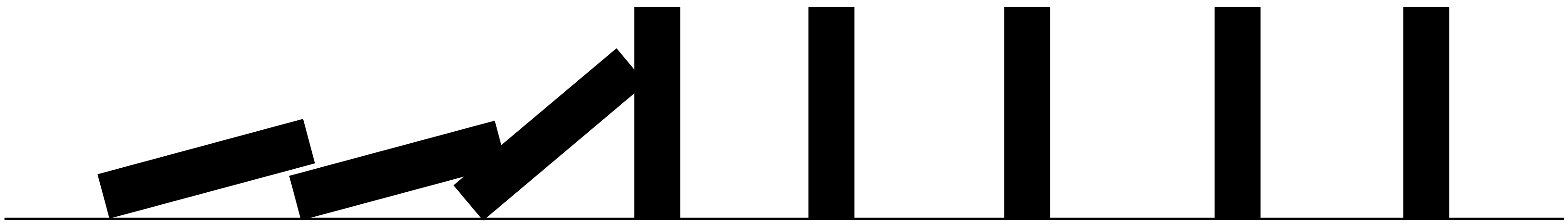

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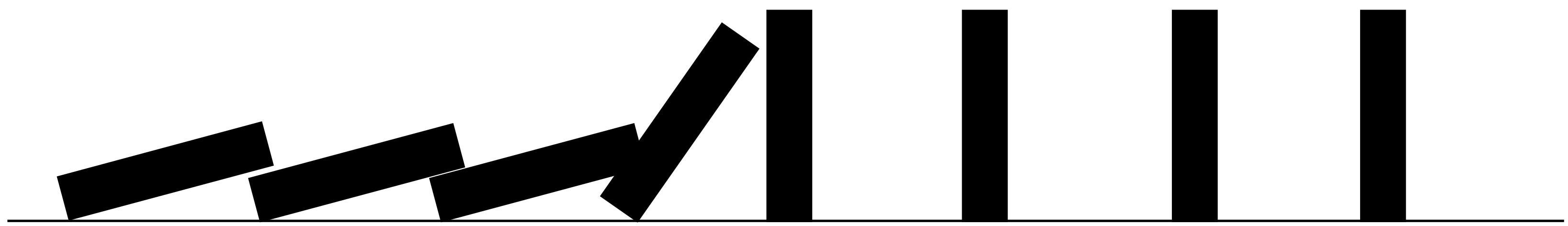

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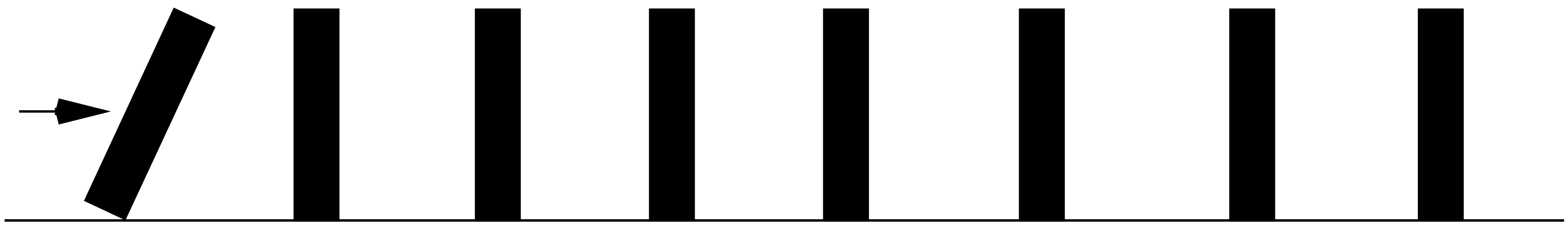

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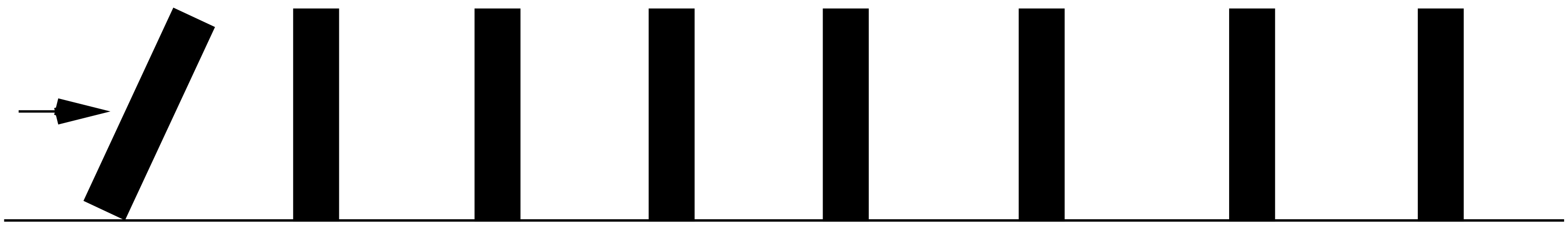
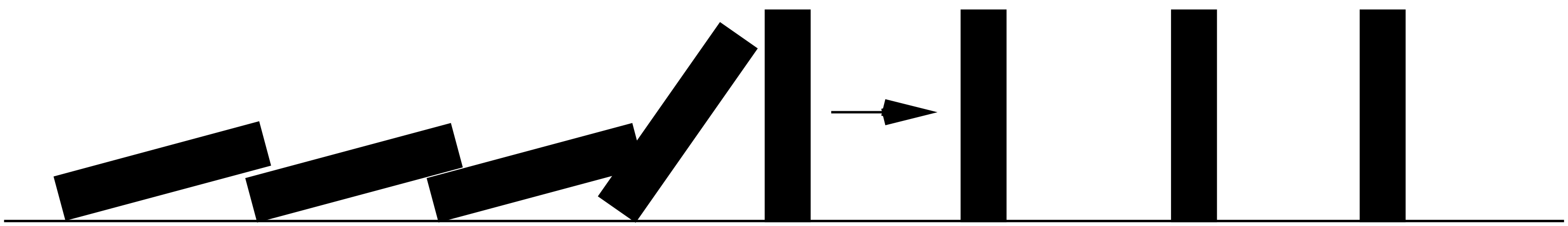


- *Each piece hits the next:*



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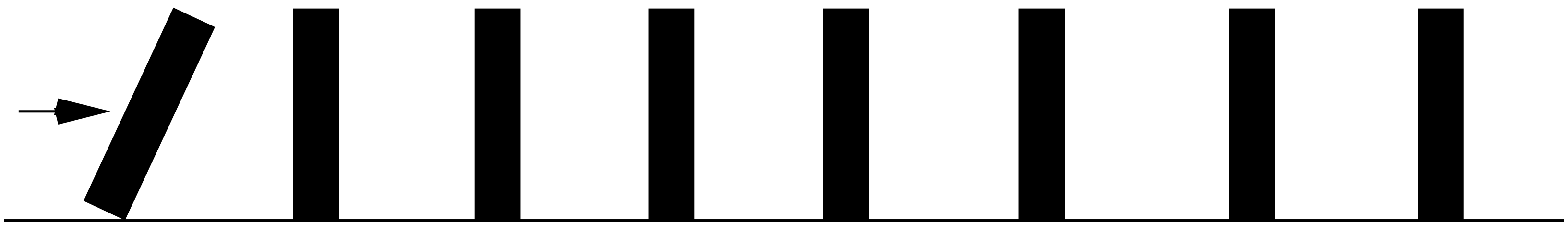

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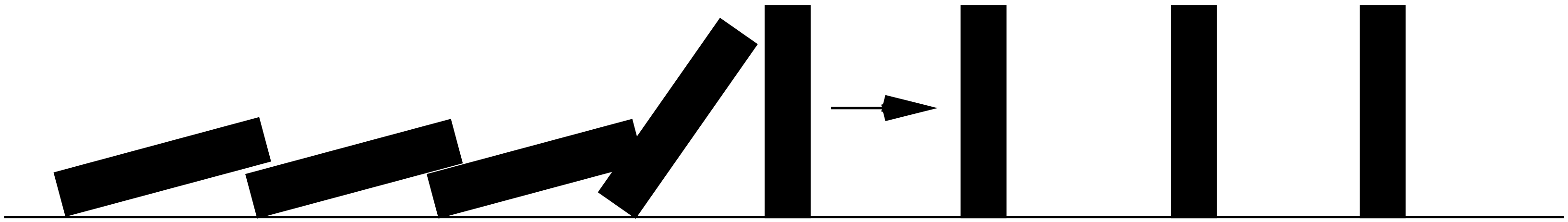

The first item is called the base case.

The second item is called the induction step.

Formally, a proof by induction proves a property $P(n)$ that depents on the positive integer n in the following way

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For every positive integer n, the sum of the first n positive integers is equal to $n(n+1)/2$.

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For every positive integer n, the sum of the first n positive integers is equal to $n(n+1)/2$.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

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What is $P(1)$*? Well, you have to substitute* $n = 1$*. Namely the sum of the first* 1 *positive integers is* $1(1+1)/2$*. In other words*

$$1 = \frac{1(1+1)}{2},$$

and this is obvious.

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We assume that the identity holds form some positive integer k, and with this hypothesis, we prove that it is true for $k+1$.

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Now we have to check the induction step, namely that

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In the language of dominoes, all the pieces have fallen up to kth, and now we check that the kth piece hits the $k+1$st.

We assume that the sum of the first k positive integers is equal to $k(k+1)/2$, and prove that the sum of the first $k+1$ positive integers is equal to ...

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Hypothesis:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Conclusion:

$$1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}.$$

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$$1+2+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}.$$

Particular case $k=100$. Then

$$1+2+\cdots+100=\frac{100(100+1)}{2}=5050$$

is assumed true, and we prove that

$$1+2+\cdots+101=\frac{101(101+1)}{2}=5151.$$

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So if you know the formula for $k = 100$, *then you get the formula for* $k = 101$. *This is how things are done in general as well.*

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$$1+2+\cdots+(k+1)=[1+2+\cdots+k]+(k+1)=$$
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The hypothesis is the formula that you obtain after substituting k *for* n*:*

$$1+2+\cdots+k=\frac{k(k+1)}{2}.$$

The conclusion is the formula that you obtain when substituting $k+1$ *for* n*:*

$$1+2+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}.$$

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In the proof we notice that the sum of the first $k+1$ *positive integers is the sum of the first* k *positive integers plus the* $k+1$*st integer. We know the formula for the sum of the first* k *positive integers, and we substitute it. Then we do the algebra, and we obtain in the end that the left-hand side is equal to the right hand side.*

Neat writing:

Problem: *Prove that for every positive integer n,*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution: *We prove this identity by induction on n.*

The base case is

$$1 = \frac{1(1+1)}{2},$$

which is true.

For the induction step we assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

and prove that

$$1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}.$$

We have

$$1+2+\cdots+(k+1)=[1+2+\cdots+k]+(k+1)=$$
$$\frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2},$$

and the induction is complete.

Of course, we know how to prove this formula by counting, but what about

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$$P(1): \quad 1^3 = \left[\frac{1(1+1)}{2}\right]^2$$

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Let us prove the induction step. Our hypothesis is that

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

Using this, we want to find the formula for

$$1^3 + 2^3 + 3^3 + \cdots + (k+1)^3.$$

Note that in this expression you have the sum of the first $k+1$ *perfect cubes. This contains the sum of the first* k *cubes, to which you add the* $k+1$*st. You can then substitute the formula for the first* k*, using the induction hypothesis.*

We therefore have

$$1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3$$

$$[1^3 + 2^3 + 3^3 + \cdots + k^3] + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = (k+1)^2 \left(\frac{k^2}{4} + k + 1\right)$$

$$= (k+1)^2 \cdot \frac{k^2 + 4k + 4}{4} = (k+1)^2 \cdot \frac{(k+2)^2}{4} = \left[\frac{(k+1)(k+2)}{2}\right]^2.$$

We have thus shown that

$$1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2,$$

which is the formula for $n = k + 1$*. The induction step is proved, and hence the induction is complete.*

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\frac{1}{2} &= \frac{1}{2} \\
\frac{1}{2}+\frac{1}{4} &= \frac{3}{4} \\
\frac{1}{2}+\frac{1}{4}+\frac{1}{8} &= \frac{7}{8} \\
\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} &= \frac{15}{16}.
\end{aligned}$$

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\frac{1}{2} &= \frac{1}{2} \\
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We guess that the denominator of the right-hand side is the same as the last denominator on the left-hand side, and the numerator is one less.

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$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

But is this guess a proof? Does this always happen to be true, or only in these cases?

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But is this guess a proof? Does this always happen to be true, or only in these cases? We need to prove this formula in the general case and we do it by induction.

Base case. *For* $n = 1$*,*

$$\frac{1}{2} = \frac{2-1}{2}$$

is true.

Induction step. *We assume that*

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}$$

and we prove that

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}.$$

We compute

$$\begin{aligned}\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{k+1}} &= \left[\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k}\right]+\frac{1}{2^{k+1}}\\ &= \frac{2^k-1}{2^k}+\frac{1}{2^{k+1}} = \frac{2(2^k-1)}{2^{k+1}}+\frac{1}{2^{k+1}}\\ &= \frac{2^{k+1}-2+1}{2^{k+1}} = \frac{2^{k+1}-1}{2^{k+1}}.\end{aligned}$$

This is the desired formula, and the induction is complete.

Show that for every positive integer n*,*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

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Now you have a sum on both sides!

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Let us try to understand first what this means. What does it say for $n = 3$? The left-hand side should end with $1/(2n) = 1/6$ and the right-hand side should run from $1/(n+1) = 1/4$ to $1/6$.

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Let us try to understand first what this means. What does it say for $n = 4$*? The left-hand side should end with* $1/(2n) = 1/8$ *and the right-hand side should run from* $1/(n+1) = 1/5$ *to* $1/8$.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}.$$

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$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}.$$

Is this true? You can check it by hand, and it is a very unpleasant computation. But it is better to just prove the formula by induction, and then you know that it is true, without having to do this unpleasant computation.

Show that for every positive integer n,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

Base case. For $n=1$

$$1-\frac{1}{2}=\frac{1}{2}$$

holds true.

Show that for every positive integer n,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

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Induction step. *We assume that*

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}=\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}$$

and prove that

$$\begin{aligned}1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2(k+1)-1}-\frac{1}{2(k+1)}\\ =\frac{1}{(k+1)+1}+\frac{1}{(k+1)+2}+\cdots+\frac{1}{2(k+1)}\end{aligned}$$

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and prove that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2(k+1)-1}-\frac{1}{2(k+1)}$$
$$=\frac{1}{(k+1)+1}+\frac{1}{(k+1)+2}+\cdots+\frac{1}{2(k+1)}$$

Let us compare the identity for $k+1$ with that for k.

Induction step. *We assume that*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}$$

and prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2(k+1)-1} - \frac{1}{2(k+1)}$$
$$= \frac{1}{(k+1)+1} + \frac{1}{(k+1)+2} + \cdots + \frac{1}{2(k+1)}$$

Let us compare the identity for $k+1$ *with that for* k*. The left-hand side has two additional terms*

$$\frac{1}{2(k+1)+1} - \frac{1}{2(k+1)}.$$

Induction step. *We assume that*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}$$

and prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2(k+1)-1} - \frac{1}{2(k+1)}$$
$$= \frac{1}{(k+1)+1} + \frac{1}{(k+1)+2} + \cdots + \frac{1}{2(k+1)}$$

Let us compare the identity for $k+1$ *with that for* k. *The right-hand side is missing a*

$$\frac{1}{k+1}$$

but has two additional terms

$$\frac{1}{2(k+1)-1} + \frac{1}{2(k+1)} = \frac{1}{2k+1} + \frac{1}{2(k+1)}.$$

Induction step. *We assume that*

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}=\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}$$

and prove that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2(k+1)-1}-\frac{1}{2(k+1)}$$
$$=\frac{1}{(k+1)+1}+\frac{1}{(k+1)+2}+\cdots+\frac{1}{2(k+1)}$$

Let us compare the identity for $k+1$ with that for k. So the right-hand side differs by

$$\frac{1}{2k+1}+\frac{1}{2(k+1)}-\frac{1}{k+1}=\frac{1}{2k+1}+\frac{1}{2(k+1)}-\frac{2}{2(k+1)}$$
$$=\frac{1}{2k+1}-\frac{1}{2(k+1)}.$$

Same thing!

So if we add

$$\frac{1}{2k+1} - \frac{1}{2(k+1)}$$

to both sides of

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k},$$

we obtain

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2(k+1)-1} - \frac{1}{2(k+1)} \\ = \frac{1}{(k+1)+1} + \frac{1}{(k+1)+2} + \cdots + \frac{1}{2(k+1)}. \end{aligned}$$

This completes the proof by induction.

Now we know that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

without having to waste our times with the computation.

HOMEWORK

1. Prove that for every positive integer n,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(Hint. This is actually solved in the book. But... try to solve it yourself without aid, and if you do not succeed, then look at the proof in the book.)

2. Prove that for every positive integer n,

$$1 + 3 + 5 + \cdots + (2n-1) = n^2,$$

in other words the sum of the first n odd integers is n^2.

3. Prove that for every positive integer n,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$